Mid-semester examination 2019
M.Math. II — Algebraic Geometry
Each question carries 10 marks.

Question 1.
(a) Show that an affine algebraic set $V \subset \mathbb{A}^n(k)$ is irreducible if and only if $I(V)$ is prime.
(b) Find the irreducible components of $V(X^3+X-X^2Y-Y)$ in $\mathbb{A}^2(\mathbb{R})$ and in $\mathbb{A}^2(\mathbb{C})$.

Question 2.
(a) Let $\varphi : \mathbb{A}^1 \longrightarrow V = V(Y^2 - X^3) \subset \mathbb{A}^2$ be defined by $\varphi(t) = (t^2, t^3)$. Show that although φ is one-to-one and onto polynomial map, φ is not an isomorphism.
(b) For the pair of ellipses $V((X-1)^2 + Y^2 - 1)$, $V(16U^2 + 9(V+2)^2 - 1)$ find a real affine coordinate change that maps the ellipse in the XY-plane to the ellipse in the UV-plane.

Question 3.
(a) Let V be a non-empty variety in $\mathbb{A}^n$ and f be a rational function on V. Show that the pole set of f is an algebraic subset of V.
(b) Let $V = V(Y^2 - X^2(X+1)) \subset \mathbb{A}^2$, and $\overline{X}, \overline{Y}$ are the residues of X, Y in $\Gamma(V)$. Let $z = \overline{X}/\overline{Y} \in k(V)$. Find the pole sets of z and z^2.

Question 4.
(a) Let $I = (Y^2-X^2, Y^2+X^2) \subset \mathbb{C}[X,Y]$. Find $V(I)$ and $dim_{\mathbb{C}}(\mathbb{C}[X,Y]/I)$.
(b) Show that the curve $\{(x,y) \in \mathbb{C}^2 \ : \ x^2+y^2-1 = 0\}$ is smooth.

Question 5.
(a) Show that every point on the double line $\{(x,y) \in \mathbb{C}^2 \ : \ (2x+3y-4)^2 = 0\}$ is singular.
(b) Find the multiple points and the tangent lines at the multiple points of the curve $-X^3 + 2X^2 + Y^2 - X - 2Y + 1$.

Question 6.
(a) Let X be an algebraic set in $\mathbb{P}^n$. Show that $I_p(X)$ is a homogeneous ideal of $k[X_0, X_1, \ldots, X_{n+1}]$.
(b) Let J be a homogeneous ideal. Prove that $V_p(J) = \varnothing$ if and only if $\langle X_0, X_1, \ldots, X_{n+1} \rangle \subseteq rad(J)$.